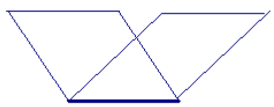
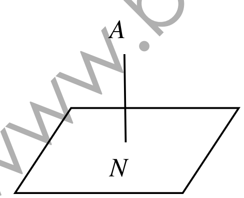
Vectors III – Planes

1	**Equation of plane** **Vector form**: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\lambda, \mu \in \mathbb{R}$ **a** is a fixed point on the plane, **b** and **c** are vectors // to the plane. Note: **b** and **c** are not // to each other. **Scalar product form** The normal(perpendicular vector) to the plane is given by: $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ Equation of plane: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ **Cartesian form** Replace **r** with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to obtain the Cartesian equation.	Find the equation of a plane that passes through $A(1,2,3)$ and is parallel to (1,1,2) and (2,3,4). Ans: $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$ -- $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, so $\mathbf{r} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ to give $\mathbf{r} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 1$ (Ans) -- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 1$ to give $-2x + z = 1$(Ans)
2	**Perpendicular distance from a point to a plane** From the scalar product form, $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ dividing both sides by the magnitude of **n**, $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}}$ The RHS $\mathbf{a} \cdot \hat{\mathbf{n}}$ represents the $\perp$ distance from the origin to the plane. Note: this formula is useful to find the *distance between 2 planes*.	$\mathbf{r} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 1$ $\mathbf{r} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{5}}$ Distance of origin to plane = $\frac{1}{\sqrt{5}}$ units
3	**Angle between 2 planes** $\cos\theta = \frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\lvert\mathbf{n}_1\rvert \lvert\mathbf{n}_2\rvert}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors of the planes Note: If **acute** angle, then $\cos\theta = \frac{\lvert\mathbf{n}_1 \bullet \mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert \lvert\mathbf{n}_2\rvert}$	
4	**Angle between 1 line 1 plane** $\sin\theta = \frac{\mathbf{n} \bullet \mathbf{b}}{\lvert\mathbf{n}\rvert \lvert\mathbf{b}\rvert}$, where $\mathbf{n}$ is normal vector of plane, **b** is direction vector of line. Note: If **acute** angle, then $\sin\theta = \frac{\lvert\mathbf{n} \bullet \mathbf{b}\rvert}{\lvert\mathbf{n}\rvert \lvert\mathbf{b}\rvert}$	

<u>5</u>	**<u>Intersection of line and plane</u>** **<u>Strategy:</u>** 1. Substitute line into plane, 2. Solve for arbitrary constant λ, 3. Sub. λ into line to find point of intersection.	Line: $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$ ----(1) Plane: $\mathbf{r}\cdot\begin{pmatrix}-2\\3\\1\end{pmatrix}=1$ ------(2) Sub. (1) into (2): $\begin{pmatrix}1+\lambda\\2+\lambda\\3+2\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\1\end{pmatrix}=1$, solve, $\lambda=-2$ Hence, pt. of intersection= $\begin{pmatrix}-1\\0\\-1\end{pmatrix}$
<u>6</u>	**<u>To find line of intersection of 2 planes</u>**  **<u>Method 1:</u>** direction of the line of intersection, **b**,is given by: $\mathbf{b}=\mathbf{n}_1\times\mathbf{n}_2$ Equation of line of intersection: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ Note: This method will not work if a point common to both planes(**a**) is not given in qn. **<u>Method 2:</u>** Use GC. (Refer to eg) **Method 3: (Manual calculation)**	**<u>Method 2</u>** Given eqns of 2 planes: $\begin{matrix}x-y+2z=3\\2x-3y-4z=5\end{matrix}$, solve with GC to get $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4-10z\\1-8z\\z\end{pmatrix}=\begin{pmatrix}4\\1\\0\end{pmatrix}+z\begin{pmatrix}-10\\-8\\1\end{pmatrix}$, which gives $\mathbf{r}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\-8\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ (Ans)
<u>7*</u>	**<u>Foot of perpendicular from a point to a plane</u>**  **<u>Strategy:</u>** 1. equation of line AN: $\mathrm{r}=\vec{OA}+\lambda\mathrm{n}$, where **n** is normal of plane 2. Substitute equation of line into plane to find λ for the point of intersection, N. 3. Substitute λ into (1) to find $\vec{ON}$, which is the foot of the perpendicular. <u>Note</u>: If need to find shortest distance from point to plane, find $\left\|\vec{AN}\right\|$.	Example: Given $A(1,2,3)$ and equation of plane $\mathrm{r}\cdot\begin{pmatrix}3\\2\\4\end{pmatrix}=4$: $l_{AN}:\mathrm{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\4\end{pmatrix}$ Substitute l_{AN} into equation of plane: $\begin{pmatrix}1+3\lambda\\2+2\lambda\\3+4\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\2\\4\end{pmatrix}=4$ solving, $\lambda=\frac{-15}{29}$, and $\vec{ON}=\begin{pmatrix}-16/29\\28/29\\27/29\end{pmatrix}$.